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On the Entropy of Quantum Fields in Black Hole Backgrounds

S.P. de Alwis[⋆]

Dept. of Physics, Box 390,
University of Colorado,
Boulder, CO 80309

N. Ohta[†]

Dept. of Physics, Osaka University,
Toyonaka, Osaka 560, Japan

ABSTRACT

We show that the partition function for a scalar field in a static spacetime background can be expressed as a functional integral in the corresponding optical space, and point out that the difference between this and the functional integral in the original metric is a Liouville type action. A general formula for the free energy is derived in the high temperature approximation and applied to various cases. In particular we find that thermodynamics in the extremal Reissner-Nordström space has extra singularities that make it ill-defined.

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⋆ e-mail: dealwis@gopika.colorado.edu
† e-mail: ohta@phys.wani.osaka-u.ac.jp

Recently much attention has been paid to the calculation of the quantum corrections [1-8] to the Bekenstein-Hawking entropy [9,10] of black holes. In this paper, we will derive a general formula for the free energy and entropy of a scalar field in an arbitrary static spacetime background in the high temperature approximation. We will show that the difference between our free energy and that in the calculations of [2,3] is due to the conformal anomaly. We will also apply our formula to various cases and in particular discuss a possible resolution of a puzzle associated with the thermodynamics of extremal Reissner-Nordström black holes.

Consider a static metric $ds^2 = g_{00}dt^2 + h_{ij}dx^i dx^j$. Writing $g = \det g_{\mu\nu} = g_{00}h,\ h = \det h_{ij}$ where $\mu,\nu = 0,...,D;\ i,j = 1,...D$, we have the action for scalar fields in this background

$$
\begin{aligned}
S &= -\frac{1}{2}\int d^D x\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
&= \int dt\int d^{D-1}x\sqrt{h}[\frac{1}{2\sqrt{-g_{00}}}\dot{\phi}^2 - \frac{\sqrt{-g_{00}}}{2}h^{ij}\partial_i\phi\partial_j\phi].
\end{aligned}
\tag{1}
$$

The canonical momentum is $\pi = \frac{\dot{\phi}}{\sqrt{-g_{00}}}$ and the Hamiltonian is

$$
H = \int d^{D-1}x\mathcal{H} = \int d^{D-1}x\sqrt{h}\sqrt{-g_{00}}[\frac{1}{2}\pi^2 + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi],
\tag{2}
$$

and the equal-time canonical commutation relations are $[\hat{\phi}(\vec{x}),\hat{\pi}(\vec{y})] = \frac{i}{\sqrt{h}}\delta(\vec{x}-\vec{y})$. By the usual time-slicing method, one finds for the partition function in this background the expression

$$
\begin{aligned}
\mathrm{Tr}[e^{-\beta H}] &= \int[d\pi]\int_{\phi(0,\vec{x})=\phi(\beta,\vec{x})}[d\phi]e^{-\int_0^\beta dt\int d^{D-1}x\sqrt{h}[-i\pi\dot{\phi}+\mathcal{H}]} \\
&= \int_{\phi(0,\vec{x})=\phi(\beta,\vec{x})}\prod_{t,\vec{x}}d\phi\left(\frac{h}{g^E_{00}}(t,\vec{x})\right)^{\frac{1}{4}}e^{-\int_0^\beta dt\int d^{D-1}x\sqrt{g^E}\frac{1}{2}g^{E,\mu\nu}\partial_\mu\phi\partial_\nu\phi}.
\end{aligned}
\tag{3}
$$

In the above $g^E_{\mu\nu} = (-g_{00},h_{ij})$ is the Euclidean metric and henceforth we will drop the superscript E. It is convenient to discuss conformally coupled scalars

and to introduce a mass term, so we will change the matter action (after partial integration) to $S_\phi = \int_0^\beta dt d^{D-1}x\sqrt{g}\phi(K+m^2)\phi$, where $K \equiv -\Box + \frac{1}{4}\frac{D-2}{D-1}R$, $\Box \equiv \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu)$. Thus we may write

$$\begin{aligned}
\mathrm{Tr}[e^{-\beta H}] &= \int_{\phi(0,\vec{x})=\phi(\beta,\vec{x})} \prod_{t,\vec{x}} d\phi\Omega g^{\frac{1}{4}}(t,\vec{x}) e^{-\int_0^\beta dt d^{D-1}x\sqrt{g}\phi(K+m^2)\phi} \\
&= \int_{\phi(0,\vec{x})=\phi(\beta,\vec{x})} \prod_{t,\vec{x}} d\phi g^{\frac{1}{4}}(t,\vec{x}) e^{-\int_0^\beta dt d^{D-1}x\sqrt{g}\phi(K+m^2)\phi + S_L[g,\Omega]}.
\end{aligned} \tag{4}$$

In the above $\Omega = \frac{1}{\sqrt{g_{00}}}$ is a conformal factor which causes a mismatch between the metric background of the action and that defining the functional integral. The effect of this term may be written as a Liouville type action. In two dimensions, it is in fact the Liouville action with the Liouville field being $\ln\Omega^2$.

Thus we have for the free energy the expression

$$-\beta F = -\frac{1}{2}\ln\det[K_\beta + m^2] + \beta\int d^{D-1}x\sqrt{g}L_L[\Omega, g]. \tag{5}$$

The second term is linear in β so that the temperature dependence of the free energy and hence the entropy ($S = \beta^2\frac{\partial F}{\partial\beta}$) comes entirely from the first term. Away from the Hawking temperature the Euclidean metric has conical singularities with $\int R \sim \beta_{Hawking} - \beta$ [11,3,2]. However these β-dependent terms vanish at the Hawking temperature and the bulk term is simply the quantum correction to the zero temperature cosmological constant* which should be canceled against a bare cosmological constant. Hence the entire free energy of the gas of particles at the Hawking temperature must come from the generalized Liouville action.

There is, however, a formulation of the path integral in which the calculation is directly related to the evaluation of the free energy of a gas of bosons. This

$\star$ This would be zero in a supersymmetric theory.

is obtained by introducing the optical metric [12]† and performing a change of field variable. Thus writing $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $\bar{\phi} = \Omega^{\frac{2-D}{2}}\phi$, we have for the measure $\prod_{t,\vec{x}} d\phi\Omega g^{\frac{1}{4}}(t,\vec{x}) = \prod_{t,\vec{x}} d\bar{\phi}\bar{g}^{\frac{1}{4}}(t,\vec{x})$. Using the properties of the Laplacian with conformal coupling under a conformal transformation (see for example [13]), we may write the partition function as

$$\begin{aligned}\mathrm{Tr}[e^{-\beta H}] &= \int\limits_{\bar{\phi}(0,\vec{x})=\bar{\phi}(\beta,\vec{x})} \prod_{t,\vec{x}} d\bar{\phi}\bar{g}^{\frac{1}{4}}(t,\vec{x}) e^{-\int_0^\beta dt d^{D-1}x\sqrt{\bar{g}}\bar{\phi}(\bar{K}+m^2\Omega^{-2})\bar{\phi}} \\ &= = -\frac{1}{2}\ln\det[\bar{K}_\beta + m^2\Omega^{-2}] = -\int\limits_\epsilon^\infty \frac{ds}{s}\int \sqrt{\bar{g}}d^D x \bar{H}(s|x,x).\end{aligned} \tag{6}$$

Here $\bar{H}(s|x,x') = e^{-s(\bar{K}+m^2\Omega^{-2})}\frac{1}{\sqrt{\bar{g}}}\delta^D(x-x')$ is the heat kernel and ϵ is an ultraviolet cutoff. Optical space has the metric $\bar{ds}^2 = dt^2 + \frac{h_{ij}}{g_{00}}dx^i dx^j$, and it has the topology $S^1 \times \mathcal{M}^{D-1}$, so that the heat kernel factorizes into that on S^1 and the one on $\mathcal{M}^{D-1}$. Hence we have the following formula for the free energy after subtracting the zero-temperature cosmological constant term (i.e. the $n=0$ term in the thermal sum):

$$F(\beta) = -\frac{1}{2}\int\limits_0^\infty \frac{ds}{s}\frac{1}{(4\pi s)^{\frac{D}{2}}}\sum_{n\neq 0} e^{-\frac{\beta^2 n^2}{4s}}\sum_{k=0}^\infty \frac{(-s)^k}{k!}\bar{B}_k. \tag{7}$$

The first factor in the integral is the heat kernel on S^1 and in the second factor we have used the well-known expansion for the heat kernel [13] with

$$\bar{B}_0 = \int\limits_{\mathcal{M}} e^{-\Omega^{-2}m^2 s}\sqrt{\bar{g}}, \quad \bar{B}_1 = (\xi - \frac{1}{6})\int\limits_{\mathcal{M}} e^{-\Omega^{-2}m^2 s}\sqrt{\bar{g}}\bar{R}, \tag{8}$$

etc., where $\xi = \frac{1}{4}\frac{D-2}{D-1}$. It should be noted that the free energy has the expected ultraviolet divergence, but it does not come from the $s=0$ end of the proper

† This metric has been used in connection with this problem also in [4,5]. But unlike in those papers here we show how this metric arises from the standard expression for the partition function.

time integral. Instead it is the divergence of the optical metric at a horizon of the original space that causes trouble. We will discuss this further by looking at particular examples, but before that let us derive a universal expression for all static spaces by using the high temperature approximation. This is easily obtained by first changing the variable of the proper time integral from s to $u = \beta^{-2}s$ and then neglecting the higher powers of β^2 coming from the expansion in (7):

$$\begin{aligned} F =& - T^D V_{D-1} \int_0^\infty \frac{du}{u} \frac{1}{(4\pi u)^{\frac{D}{2}}} \sum_{n=1}^\infty e^{-\frac{n^2}{4u}} \\ =& - \frac{T^D V_{D-1}}{\pi^{\frac{D}{2}}} \Gamma\left(\frac{D}{2}\right) \zeta(D), \end{aligned} \tag{9}$$

where $V_{D-1} = \int_{\mathcal{M}_{\mathrm{D-1}}} \sqrt{g}$ is the volume of optical space. This is just the free energy of a gas of (massless) particles in a box whose volume is given by the optical measure. Thus in four dimensions we have

$$F = -V_3 T^4 \frac{\pi^2}{90}. \tag{10}$$

Note that these formulae for the free energy are physically relevant only at the Hawking temperature but we need these expressions at arbitrary T to calculate the entropy of the quantum fields from the relation $S = -\frac{\partial F}{\partial T}|_{T=T_H}$.

Let us now discuss some examples. The first is 2D Rindler space. The entropy has been calculated by several authors [2,3] using the path integral in the original metric (4), but the Liouville action term of this equation was not kept, so that the free energy at the Hawking temperature was not obtained by them. Let us check that this term indeed gives the right value for the free energy of massless particles. Euclidean Rindler space has the metric $ds^2 = R^2 d\omega^2 + dR^2$ and the relevant conformal factor in (4) is $\Omega = \frac{1}{R}$. The Hawking temperature T_H is $\frac{1}{2\pi}$, so

at this value the free energy is given by the Liouville action. Thus we have⋆

$$-2\pi F = S_L = \frac{1}{24\pi}\int d^2x\sqrt{g}g^{\mu\nu}\partial_\mu \ln\Omega\partial_\nu \ln\Omega = \frac{1}{24\pi}\int\limits_0^{2\pi}\int\limits_\epsilon^L RdR\frac{1}{R^2}, \tag{11}$$

and the free energy is $F = -\frac{1}{24\pi}\ln\frac{L}{\epsilon}$. This agrees with what one gets by using the optical metric formulation in which case one has the exact result (9) (since all curvature terms are zero) that the free energy is that of a gas of (massless) bosons in a box of optical volume $V_1 = \int_\epsilon^L \frac{dR}{R}$ at $T_H = \frac{1}{2\pi}$. In dimensions greater than two, however, the optical curvature is non-zero ($\bar{\mathcal{R}} = -(D-1)(D-2)$) and one has to use the high temperature approximation, i.e. (9) with $V_{D-1} = V_{D-2}\int_\epsilon^\infty \frac{dR}{R^{D-1}} = \frac{V_{D-2}}{(D-2)\epsilon^{D-2}}$. Thus in four dimensions we find (using (10)) $F = -\frac{A}{\epsilon^2}T^4\frac{\pi^2}{180}$, where A is the transverse area in agreement with the first calculation of [3].

We will now discuss (four-dimensional) black hole spaces. The Schwarzschild metric (setting $G_N = 1$) is $ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{(1-\frac{2M}{r})} + r^2 d\Omega_2$ and the corresponding optical volume is

$$\begin{aligned} V_3^{Sch} &= 4\pi\int\limits_{2M+\epsilon}^R \frac{r^2}{(1-\frac{2M}{r})^2}dr \\ &= 4\pi[\frac{R^3}{3} + 2MR^2 + 12M^2R + 32M^3\ln\frac{R-2M}{\epsilon} \\ &+ \frac{16M^4}{\epsilon} - \frac{104M^3}{3} + O(R^{-1}) + O(\epsilon)]. \end{aligned} \tag{12}$$

By plugging this into (9), we immediately get the free energy and hence the entropy of a scalar field in a black hole background. Here we see the divergence first observed by [1]. Although it appears linear in terms of the coordinate cutoff ϵ, it is quadratic in terms of the proper distance cutoff $\delta = \sqrt{2M\epsilon}$ in the Schwarzschild geometry. We also see another logarithmic divergence. These additional divergences can also be found [7] by working with the functional integral in the original metric (4). However in that case the calculation is much more complicated.

⋆ For a related calculation see [14].

Next let us consider the Reissner-Nordström charged black hole. This example is interesting because it has an extremal limit when the mass becomes equal to the charge. The metric is

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2})dt^2 + (1 - \frac{2M}{r} + \frac{Q^2}{r^2})^{-1}dr^2 + r^2 d\Omega_2. \tag{13}$$

This black hole has an ADM mass M and an electric charge Q. The metric has outer and inner horizons at $r_\pm = M \pm (M^2 - Q^2)^{\frac{1}{2}}$. In order to avoid a naked singularity we must have $M \geq Q$. The Hawking temperature of this hole is given by $T = \frac{(r_+ - r_-)}{4\pi r_+^2}$ (which goes to zero as $M \rightarrow Q$ and the entropy is again given by the quarter of the area of the horizon $S = \frac{1}{4} 4\pi(2M)^2 = 4\pi M^2$ as in the Schwarzschild case. In the limit $M \rightarrow Q$, the two horizons become degenerate and the metric of this extremal hole is

$$ds^2 = -(1 - \frac{M}{r})^2 dt^2 + \frac{dr^2}{(1 - \frac{M}{r})^2} + r^2 d\Omega_2. \tag{14}$$

Although the limiting temperature of the RN black hole in the extremal limit is zero, purely geometrical considerations of the extremal hole metric itself indicate that the temperature of this extremal hole is arbitrary and that its entropy is zero [15] even though the area of the horizon is non-zero. This seems to be rather puzzling from the thermodynamic point of view.[⋆] We shall see below that the calculation of the contribution of the scalar fields to the entropy sheds some light on this issue.

⋆ We wish to thank L. Susskind for pointing this out.

For the non-degenerate case, the optical volume is

$$
\begin{aligned}
V_3^{rn} =& 4\pi \int_{r_+ + \epsilon}^{R} \frac{r^6 dr}{(r-r_+)^2(r-r_-)^2} = 4\pi[\frac{R^3}{3} + 2MR^2 + (3r_+^2 + 4r_+r_- + 3r_-^2)R \\
&+ \frac{r_+^6}{(r_+-r_-)^2\epsilon} + \frac{r_-^6}{(r_+-r_-)^3} + \frac{2r_+^5(2r_+-3r_-)}{(r_+-r_-)^3}\ln\frac{R-r_+}{\epsilon} \\
&+ \frac{2r_-^5(3r_+-2r_-)}{(r_+-r_-)^3}\ln\frac{R-r_-}{r_+-r_-} - r_+(\frac{13}{3}r_+^2 + 5r_+r_- + 3r_-^2) \\
&+ O(R^{-1}) + O(\epsilon)].
\end{aligned} \tag{15}
$$

Substituting this in (10), we have the expressions for the (quantum corrections to the) free energy and hence also the entropy in this space. The leading divergence is again linear (or quadratic in the proper cutoff) and there is an additional logarithmic divergence. However we also see the appearance of inverse powers of the difference in the two horizon radii. This clearly implies that the extremal limit is very singular. Indeed this is confirmed by a direct calculation of the extremal black hole free energy and entropy.

From (14) we have for the optical volume

$$
\begin{aligned}
V_3^{ext} = \int_{M+\epsilon}^{R} \frac{r^6 dr}{(r-M)^4} = 4\pi[&\frac{R^3}{3} + 2MR^2 + 10M^2R + \frac{M^6}{3\epsilon^3} + \frac{3M^5}{\epsilon^2} \\
&+ \frac{15M^4}{\epsilon} + 20M^3\ln\frac{R-M}{\epsilon} - \frac{37}{3}M^3 + O(R^{-1}) + O(\epsilon)].
\end{aligned} \tag{16}
$$

Here we see the appearance of cubic and quadratic divergences. Clearly the thermodynamics of the extremal limit is not well-defined since although the linear and logarithmic divergences may be absorbed into the renormalization of G_N [3] and the coefficients of higher powers of curvature in the expansion of the effective action, this will not be the case for these higher order divergences. *The point is that in the limit $M \to Q$ the temperature $T_H \simeq (r_+ - r_-) \to 0$ so that the free energy (see* (10) *) goes to zero while the entropy correction is logarithmically divergent.*

However in the extremal case (14) *the temperature is arbitrary [15] so that both the free energy and the entropy will diverge cubically.* This suggests therefore that the thermodynamics of the extremal RN black hole (14) as opposed to the limiting case of the RN black hole (13) is not well-defined.

Our last example is the dilaton black hole [16]*. The metric is given in this case by

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r(r-a)d\Omega_2, \tag{17}$$

where a is a constant. The corresponding optical volume is

$$\begin{aligned} V_3 &= 4\pi \int\limits_{2M+\epsilon}^{R} \frac{r(r-a)}{(1-\frac{2M}{r})^2} dr \\ &= 4\pi[\frac{R^3}{3} + (2M - \frac{a}{2})R^2 + 4M(3M-a)R + \frac{8M^3(2M-a)}{\epsilon} \\ &+ 4M^2(8M-3a)\ln\frac{R-2M}{\epsilon} - M^2(\frac{104M}{3} - 10a) + O(R^{-1}) + O(\epsilon)]. \end{aligned} \tag{18}$$

As in the Schwarzschild case, here too there are linear as well as logarithmic divergences and again one may argue following [3] that the former can be absorbed in a renormalization of G_N. In the extremal limit ($M \rightarrow \frac{a}{2}$), the "classical" entropy ($S_{cl} = \frac{A}{4} = 2\pi M(2M-a)$ [17]) vanishes and so does the linear divergence. However the logarithmic divergence remains.

Finally let us point out that our thermodynamic entropy calculation has a bulk contribution in all finite mass black hole cases. Thus (unlike in Rindler space [3,2]) this cannot be identified with the microscopic entropy which is expected to be proportional to the area of the horizon [18,19].

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$\star$ This case has been discussed using a different method in [8].

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